Boundary element and finite element analysis for the efficient simulation of fluid-structure interaction and its application to mold filling processes

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Abstract: Metal casting and polymer molding are widely used for the economical shape processing of complex geometries. In these manufacturing processes, a liquid melt (metal, mineral or synthetic) is filled into a mold with a cavity of the desired shape. Cooling and solidification of the melt results in a product with almost the same shape as the cavity. Numerical simulations can be employed to increase the accuracy of the process. To this end, a boundary element method for Stokes flow and a finite element formulation for liquid membranes are investigated in this work.

1 Introduction

The boundary element method (BEM) is suitable to solve partial differential equations (PDE). The BEM reduces the dimension of a problem, i.e. boundary discretization and integration is sufficient to solve a domain problem. A necessary condition for the applicability of BEM is the existence of a Greens's function for the particular PDE. For instance, BEM can solve Laplace's and Poisson's equation (heat conduction, electrostatics and potential flow), the Helmholtz equation (acoustic, electromagnetic and fluid waves) and the biharmonic equation (linear elasticity and Stokes flow). We focus on BEM for Stokes flow to model the flow within the melt. Under the assumption of low Reynolds numbers, which is valid for creeping flow, the inertial fluid forces are negligible. Therefore, the Naver-Stokes equations reduce to the linear Stokes equations.
An efficient finite element (FE) formulation for liquid membranes has been presented by Sauer et al. (2014). Here it is used to model the deformation of the melt surface itself as well as the interaction between melt and cavity. Volume discretization is also avoided in Sauer et al. (2014). For both methods, the computational as well as the meshing effort is highly reduced compared to volumetric FE.

2 FE formulation for liquid membranes

We focus on pure membranes that do not resist bending and out-of-plane shear. An FE-formulation for those membranes can be found in Sauer et al. (2014). The authors present a membrane constitutive model that models isotropic surface tension and is therefore suitable to describe liquid membranes under hydrostatic conditions. Mechanical contact at the membrane surface can be modeled by enforcing a contact constrain and considering contact contributions to the surface force. A common approach to solve the constrained optimization problem numerically is the penalty method which is used in Sec. 5.

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3 BEM for three-dimensional Stokes flow

In creeping fluid flow (Reynolds number $Re \ll 1$) convective inertial forces are negligible small compared to viscous forces. Further assuming small accelerations, the motion of an incompressible Newtonian fluid in domain Ω with boundary Γ can be described by the steady Stokes equation

$$-\nabla p + \eta\,\nabla^2 \boldsymbol{v} + \rho\,\boldsymbol{b} = \boldsymbol{0} \quad \text{in}\,\Omega, \tag{1}$$

where pressure and velocity of the creeping fluid are denoted by p respectively $\boldsymbol{v}$, while η denotes its viscosity and ρ its mass density and $\boldsymbol{b}$ denotes a body force. (1) can be transformed into the boundary integral equation (BIE) (see e.g. Pozrikidis (2002))

$$c(\boldsymbol{x})\,v_i(\boldsymbol{x}) + \frac{1}{8\,\pi}\int_\Gamma T_{ijk}(\boldsymbol{x}-\boldsymbol{y})\,v_j(\boldsymbol{y})\,n_k(\boldsymbol{y})\,\mathrm{d}\Gamma_{\boldsymbol{y}} - \frac{1}{8\,\pi\,\eta}\int_\Gamma G_{ij}(\boldsymbol{x}-\boldsymbol{y})\,t_j(\boldsymbol{y})\,\mathrm{d}\Gamma_{\boldsymbol{y}} = 0, \tag{2}$$

where summation is implied over j and k. $\boldsymbol{n}$ denotes the outward unit normal, while $\boldsymbol{t}$ denotes the boundary traction $\boldsymbol{t} = \boldsymbol{\sigma}\,\boldsymbol{n}$ with stress tensor $\boldsymbol{\sigma}$. The Green's function for velocity and traction can be obtained by

$$G_{ij}(\hat{\boldsymbol{x}}) = \frac{\hat{x}_i\,\hat{x}_j}{r^3} + \frac{\delta_{ij}}{r} \qquad T_{ij}(\hat{\boldsymbol{x}}) = -6\,\frac{\hat{x}_i\,\hat{x}_j\,\hat{x}_k}{r^5}, \tag{3}$$

with $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{y}$ and $r = \|\hat{\boldsymbol{x}}\|$. The face factor c is chosen as the fraction of the solid angle. For $x \in \Gamma$ of class C^1, it yields $c = 0.5$. Green's function $\boldsymbol{G}$ and $\boldsymbol{T}$ are singular for $r = 0$ which requires special care with respect to integration

4 Numerical validation

The FE-formulation for liquid membranes (Sauer et al., 2014) has been sufficiently validated in several applications, including liquid-solid contact (e.g. Sauer et al. (2014); Sauer (2014, 2016)). For the sake of brevity, we focus on the validation of three-dimensional BEM for Stokes flow here. We investigate two simple test cases, whose analytic solutions are available in the literature (e.g. Chwang and Wu (1975)). **Test case 1:** A solid sphere (radius $R = L_0$) is surrounded by a fluid of velocity $\boldsymbol{v} = v_0\,[\cos\theta,\, \sin\theta,\, 0]^\mathsf{T}$ with $\theta = \frac{4}{9}\,\pi$. **Test case 2:** Flow between two spheres ($R_\text{in} = L_0$, $R_\text{out} = 2\,L_0$) that are rotating in opposite direction around the z-axis with angular velocities $\boldsymbol{\omega}_\text{in} = \omega_0\,[0,\, 0,\, 1]^\mathsf{T}$ and $\boldsymbol{\omega}_\text{out} = -\,\boldsymbol{\omega}_\text{in}$. A bi-quadratic NURBS patch is used to discretize a sphere efficiently (number of control points $n = 9$). Evaluating the BIE (2) at a single point leads to $d = 3$ equations with $d\,n$ unknowns. To obtain a system of linear equations with the same number of equations and unknowns, the BIE is collocated at n collocation points $\boldsymbol{x}_a \in \Gamma$ for $a \in \{1, .., n\}$. Collocation points are chosen according to the Greville absciss sae (see e.g. Greville (1964)). Due to the continuity of the surface, the face factor yields $c(\boldsymbol{x}_a) = 0.5$. Numerical integration is carried out by using standard Gauss-Legendre quadrature on integration elements. These elements are obtained by splitting the NURBS knot spans at the values of the Greville abscissae. Thus, Green's functions (3) are only singular at the boundaries of these elements. While the number of elements is kept constant, the number of quadrature points is systematically increased and the resulting drag (test case 1) respectively moment (test case 2) is compared to analytical solutions. Fig. 1(a-b) show the relative error of the particular solution against the total number of quadrature points, while the velocity component $\boldsymbol{v}_y$ in the x-z plane is depicted in Fig. 1(c) for test case 2. For both test cases, the numerical result converges to the analytical solution with a constant rate. Nevertheless, a rather large number of quadrature points are required to obtain very accurate solutions. Special quadrature rules for singular integration might increase the computational efficiency.

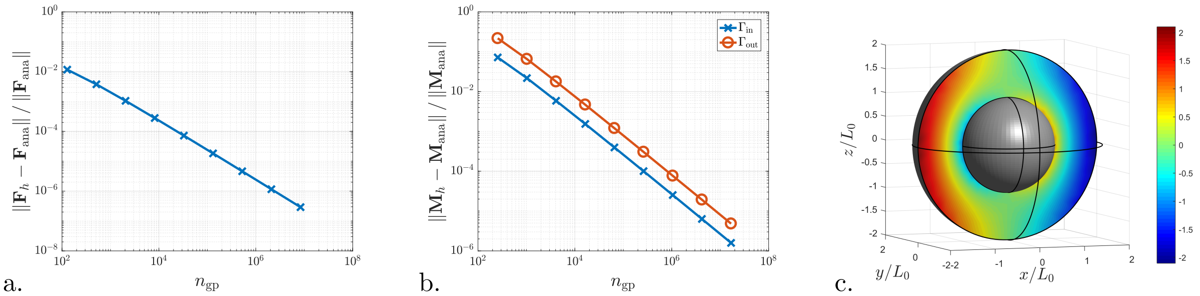

Figure 1: Validation of BEM for Stokes Flow: test case 1: Relative error of drag $\mathbf{F}$ (a), test case 2: relative error of moment $\mathbf{M}$ (b) and velocity $\boldsymbol{v}_y$ in the x-z plane (c)

5 Application to mold filling processes

FE membranes and BEM for Stokes flow can be used to model mold filling processes efficiently. Here, the deformation of a liquid melt under hydrostatic conditions is solved with the FE formulation from Sauer et al. (2014). The mechanical contact between melt and cavity is incorporated with the penalty method. The resulting velocity field can then be determined with the BEM where FE deformations serve as Dirichlet boundary conditions. The magnitude of velocity at proceeding time steps is shown in Fig. 2.

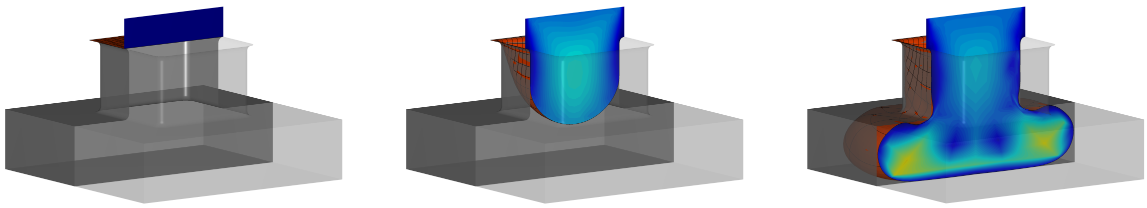

Figure 2: Application to a mold filling process: magnitude of velocity $\|\boldsymbol{v}\|$ at proceeding time steps of a filling process

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